Title	FLUID MOTION AND HEAT MOTION IN CLASSICAL FLUID DYNAMICS (Recent Advances on Mathematical Aspects of Nonlinear Wave Phenomena)
Author(s)	増田, 茂
Citation	数理解析研究所講究録 (2016), 1989: 161-169
Issue Date	2016-04
URL	http://hdl.handle.net/2433/224560
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

数理解析研究所講究録
第 1989 巻 2016 年 161-169

FLUID MOTION AND HEAT MOTION IN CLASSICAL FLUID DYNAMICS

流体数理古典理論研究所 増田 茂
SHIGERU MASUDA
RESEARCH WORKSHOP ON CLASSICAL FLUID DYNAMICS,
EX. LONG-TERM RESEARCHER, INSTITUTE OF MATHEMATICAL SCIENCES, KYOTO UNIV.

ABSTRACT.
We discuss the historical wave theories including the fluid dynamics, the heat theory, in particular, that of Fourier and Poisson. We think, from the heat theories, we have had the many mathematically fruitful productions of derivatives from the heat theories, such as the trigonometric series, the eigenvalue problems and the rapidly decreasing function.

Mathematics Subject Classification 2010 : 01Axx, 76A02, 76Mxx, 76-02, 76-03, 33A15, 35Qxx 35-xx, 35J10.

Key words. The Navier-Stokes equations, fluid dynamics, fluid mechanics, hydrostatics, hydrodynamics, hydromechanics, thermodynamics, heat difusion equations, wave equations, mathematical history, physico-mathematics, trigonometric series, Fourier series.

1. INTRODICTION

[1,2,3] Fourier explains the motion of the heat in the interior of solid. The difference is that determines its increment of the temperature during an instant :

$$Kdydz\ d\Big(\frac{dv}{dx}\Big)dt + Kdxdz\ d\Big(\frac{dv}{dy}\Big)dt + Kdxdy\ d\Big(\frac{dv}{dz}\Big)dt \quad \Rightarrow \quad Kdxdydz\Big(\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}\Big)dt$$

$$(d)_{F2.5} \qquad \frac{du}{dt} = \frac{K}{C.D}\left(\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}\right) \tag{1}$$

where, K internal conductibility, C capacity, D density of the substance. ([3, pp.120-2]). We think that Fourier's deductive method is very diffuse style and simpler than Poisson's inductive method described over 10 pages in original [16], we show his point below in § 3.1.

1.1. Poisson's paradigm and singularity.

Poisson publishes the last books consist of three elements : [13, 14, 15, 16]. ([14, 15] are the same title and are divided into two volumes.) These are his paradigm of the mathematical physics through all his academic life, entitled a study of mathematical phisics. (*Un Traité de Physique Mathématique.*) In the rivalry to Euler, Lagrange, Laplace, Fourier, Navier, et al.,

Date: 2016/01/12.
[1]Translation from Latin/French/German into English mine, except for Boltzmann.
[2]To establish a time line of these contributor, we list for easy reference the year of their birth and death: Newton (1643-1727), Euler(1707-83), d'Alembert(1717-83), Lagrange(1736-1813), Laplace(1749-1827), Fourier(1768-1830), Poisson(1781-1840), Cauchy(1789-1857), Dirichlet(1805-59), Riemann(1826-66), Boltzmann(1844-1906), Hilbert(1862-1943), Schrödinger(1887-1961).
[3]We use ($\Downarrow$) means our remark not original, when we want to avoid the confusions between our opinion and sic. ($\Leftarrow$) means our translation in citing the origin.

we think, he struggles to make his paradigm. On the other hand, as its proofs, there are some singular but important sugestions such as :

- rigorous sum instead of integral, (cf. §1.2)
- critics to easy applying the rule comes from real to transcendental function, (cf. fig.1)
- conjecture on the defect of the proof in the eternity of exact differential,
- contribution to the fluid dynamics, especially, to the Navier-Stokes equations, (cf. [7, pp. 261-271], Table 2, and 3)
- deduction of another heat equation from the basically molecular analysis. (cf. § 3 and § 3.1)

1.2. **A comment on continuum by Duhamel.**
Duhamel 1829 [1] points out the theory of continuum from the viewpoint of scientific history, citing from the Poisson's paper in the argument with Navier on the nonsense of Navier's null action in nature.

> ($\Leftarrow$) Up to now, the reserchers have considered the corps of the nature as continue, it makes illusion to this regards, however, partly because this hypothesis simplify the calcul, and partly because they think that it gives a sufficient approximation. Mr. Poisson think that this hypothesis isn't never admissible, and justify his opinion with following considerations.

> In this state, the distance which separate the molecules must be such that this condition were replaced, in having regard to their mutual attraction and the caloric repulsion which we take also among the molecular actions. However the corps is hard or something solid, the force which opposes the separation of their parties is zero or doesn't exist in the state of which we discuss. It doesn't begin the existence that when we seek to effectuate this separation, and when we change only a few distance of the molecules. Namely, *if we explain this force with a integral, it gets to as its value being zero in the natural state of corps, it will be so even if after the variation of the molecular distances, so that, the corps will opposite any resistance to the separatiopn of its parties ; this is what will be nonsense.* It results from here, that the sum which explain the total action of a series of disjoint molecules can't convert the sum instead of the definite integral ; this is what holds in the nature of the *function of distances* which represent the action of each molecule. The molecular force, of which we will find the expression in the §1 of this Memoire, is calculated according to this principle, and reduced at least in the simplest form of which it were susceptible. [1, pp.98-99] (trans. and italics mine.)

2. Confusions and unify on continuum theory

The hysico-mathematicians are must construct at first the physical structure, then allpies the mathematical concept on it. The former is necessary to fit with the actual phenomena. Arago 1829 [?] seeks to separate these items to Navier 1829 [8] in the current of dispute with Poisson and Arago. This is comes from the word what-Navier-called *l'une sur l'autre*, he fails to explain exactly it, and since then, his theories and the equations are neglected up to the top of the 20th century. We consider that the confusions and unify are as follows :

- Poisson and Fourier discuss on the handling of the De Gua's theorem into the transcendental equations. Without clear explanation, Fourier passed away in 1830. cf. (fig.1)

- On the attraction and replusion of molecule, Navier depends on Fourier's principle of heat molecule. The then hysico-mathematicians had little evaluated Navier until the top of the 20th century. For formulation of heat motion in the fluid, Fourier cites not Navier's fluid equations, but Euler's fluid equations.
- The hydrodynamists like Navier, Poisson, Cauchy are propose the wave equations in the elasticity, and the last two hydrodynamists proposes the total equations in unity on the continuum.
- On the formulation of heat motion in the fluid, Fourier had submitted this paper, however, until his death, he has not published it, in which he seems to aim the unity of hydro- and thermodynamics, however, he has given up it.

We show the difference between Poison and Fourier in applying the rule of De Gua, which is reduced for real to the transcendental. Fourier shows only real case in $a = 2,\ b = 1$ in Poisson's.

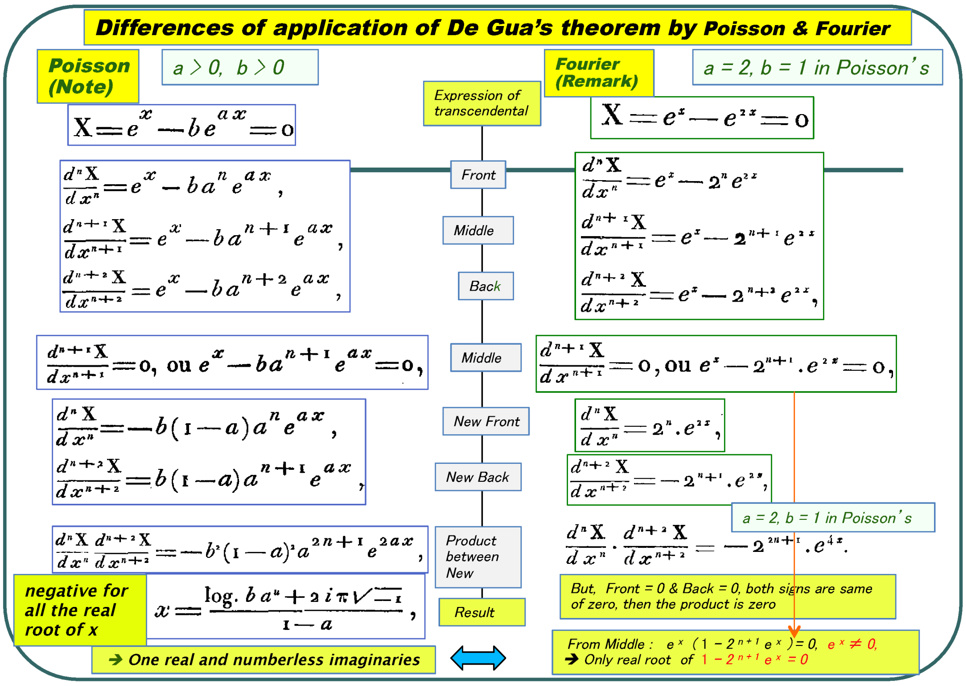

fig.1 Difference of applying the De Gua's theorem into the transcendental between Poisson and Fourier

3. The heat and fluid theories in the 19th century

Poisson [11] traces Fourier's work of heat theory, from the another point of view. Poisson emphasizes, in the head paragraph of his paper, that although he totally takes the different approaches to formulate the heat differential equations or to solove the various problems or to deduce the solutions from them, the results by Poisson are coincident with Fourier's.

> I will take care of, through this Memoire, to cite the principle result which Mr. Fourier have obtained before me ; and I dare to say at first, in all the particular problems which we have taken the one and the another for examples, and which being naturally indicated in this material, the formulae of my Memoire coincides with that this piece includes. However, *just only that there is common between our two oeuvres* ; because, it were to formulate the differential equations of the motion of the heat, or it were to solve them and deduce the definitive

solution of each problem, *I am using the entirely different methods from that Mr. Fourier is tracing.* [11, pp.1-2] (trans. and italics mine.)

3.1. The deduction of hrat equations by Poisson.

Poisson deduces his heat equations of the motion in interior of solid corps or liquid with the function R, which depend on the distance between the two molecules.

(§44.) There is always the heat in motion in all the corps, even when of all their points is invariable,

- were each point would have a particular temperature,
- were its would have all a same temperature.

However, the expression *motion of the heat* is taken here, in the another sense ; it signifies the variation of temperature which holds from an instant to the other in a corps which is heated or is cooled ; and the velocity of this motion, in each point of the corps, is the primary differential coefficient of the temperature with respect to the time.

I will call A the corps solid or liquid, homogeneous or heterogeneous, in which we are going to consider the motion of the heat. Let

- M a certain point of A,
- and m a particle of this corps, of insensible magnitude (no. 7),
- and take the point M.

At the end of a certain time t,

- designate with $x,\ y,\ z$, the three rectangular coordinates of M,
- with v the volume of m,
- and with ρ its density,

so that we have $m = v\rho$. Let also, at the same instant, u the temperature and $\mho$ [4] the velocity of motion of the heat which responds to the point M.

The quantity u will be a function of $t,\ x,\ y,\ z$, dependent on an equation in the partial differences with respect to these four variables, which it is the problem to form. If A is a corps solid, and which we make neglect its small dilations, positive or negative, products with the variations of u relative to time, the coordinates $x,\ y,\ z$, according to independent of t, and we will have simply, $\mho = \frac{du}{dt}$.

- If in contrast, we have regard to small displacement of the point M caused from these dilations,
- or also, if A is a fluid in which the integrality of temperature, or all other cause, hold to the motions of its molecules,

then the coordinates $x,\ y,\ z$, will be the function of t ; and then we will have with the known rules of the differentiation of functions made of functions,[5]

$$(1)_{PS4} \qquad \mho = \frac{du}{dt} + \frac{dx}{dt}\frac{du}{dx} + \frac{dy}{dt}\frac{du}{dy} + \frac{dz}{dt}\frac{du}{dz} \ ; \qquad (2)$$

where, expression in which $\frac{dx}{dt},\ \frac{dy}{dt},\ \frac{dz}{dt}$, will be the components of the velocities at the point M, parallel to the axes $x,\ y,\ z$. (Below, original citation omitted.)

(§45.) Let M' a second point of A very near to M, and m' a particle of A of insensible magnitude, like m which will contain M'. At the end of time t, we call $x',\ y',\ z'$, the coordinates of M' in relating to same axes with $x,\ y,\ z$, and designate with u' the temperature of m' ; also let r the distance MM'.

According to the general hypothesis on which the mathematical theory of the heat (no. 7) is based, there will be a continuous exchange of heat between m and m'. I will represent with δ

[4]($\Downarrow$) We use $\mho$, because, in origin, Poisson uses the vertical type of $\propto$ like the opened shape in upper of the numerical letter 8, however, this exact type isn't in our LaTex font system.

[5]($\Downarrow$) sic. The function is repeated.

the augmentation of heat which will result then for m during the instant dt, namely, the excess positive or negative, during this instant,

- of the heat emitted from m' and absorbed with m,
- over the heat emitted from m and absorbed with m'.

It will be able to suppose this excess proportional to product $m\ m'\ dt$, or to $v\ v'\ \rho\ \rho' dt$, in calling v' and ρ' the volume and the density of m', so that we would have $m' = v'\ \rho'$, as we have already $m = v\ \rho$. It will be zero in the case of $u' = u$, and same sign with the difference $u' - u$, when it won't be zero ; in the vacuum, it will come in the reverse ratio of the square of r ; and generally its value will be the form

$$(2)_{PS4} \qquad \delta = \frac{v\ v'}{r^2}\ R\ (u' - u)\ dt, \tag{3}$$

where, in designating with R a positive coefficient, in which we contain the factor $\rho\ \rho'$, which will decrease very rapidly for the values increasing with r, which will be also able to depend on materials and the temperatures of m and m', and will vary with the direction of MM', when the absorption of the heat won't be the same in all direction around of M.

In the supposition the most general, R will be hence a function of $r,\ u,\ u'$, and the coordinates of M and M' ; so that we will have

$$R = \Phi\ (r,\ u,\ u',\ x,\ y,\ z,\ x',\ y',\ z').$$

(Below, original citation omitted.)
(§46.) The total augmentation of heat of m during the instant dt will be the sum of values of δ, extended to all the point M' of which the distance at the point M is smaller than l. I will indicate a such sum with the characteristic Σ. The factor $v\ dt$ being common to all the value of δ, their sum will be

$$v\ dt \sum \frac{R}{r^2}\ (u' - u)\ v'. \tag{4}$$

However, during the instance dt, the temperature of m augments with $\mho\ dt$; if hence, we call c its specific heat, $c\ v\ \mho\ dt$ will be also its augmentation of heat during this instant ; hence in suppressing the common factor $v\ dt$, we will have

$$(3)_{PS4} \qquad c\ \mho = \sum \frac{R}{r^2}\ (u' - u)\ v'. \tag{5}$$

for the equation of motion of the heat equally applicable to a corps solid and to a liquid, in substituting the convenient expression with $\mho$. (Below, original citation omitted.)
(§47.) Of the point M as center and a radius equal to the linear unit, we describe a spherical surface ; were ds the differential element of this surface, to which gets, the radius of which the direction is that of MM', we will have $dv' = r^2\ dr\ ds$; and according to the value of the sum $\sum$, the equation (5) will turn out

$$(4)_{PS4} \qquad c\ \frac{du}{dt} = \iint R\ (u' - u)\ dr\ ds\ ; \tag{6}$$

We put here, for abridgement, $\frac{du}{dt}$, instead of $\mho$; however, we will remember that this differential coefficient needs to be taken with relation to t and to all this that depend ; so that it needs to replace $\frac{du}{dt}$ with the formula (2), when the coordinates $x,\ y,\ z$, of the point M will vary with the time.

The limit relative to r of the integral contains in this equation (6) won't be the same, according to the distance of the point M to the surface of A will surpass l or will be shorter than this small segment. In this chapter we will suppose that this were the primary case which holds ; the integral relative to r will come to be hence taken from $r = 0$ to $r = l$, in all the direction around M ; we will be able hence to describe the equation (6) under the form

$$(5)_{PS4} \qquad c\,\frac{du}{dt} = \int_0^l \left[\int R\,(u'-u)\,ds\right] dr\ ; \tag{7}$$

where, the integral in respecting to ds will come to be extended to all the element ds from the spherical surface, and with the reduction in series, we will obtain easily the approximate value.

(§48.) For these things, I designate with $\alpha,\ \beta,\ \gamma$, the angles which the segment MM' makes with the parallels to the axes $x,\ y,\ z$, traced through the point M. Because of $MM' = r$, then it will result

$$x' - x = r\ \cos\ \alpha, \qquad y' - y = r\ \cos\ \beta, \qquad z' - z = r\ \cos\ \gamma\ ;$$

and, according to the theory of Taylor, we will have

$$\begin{aligned} u' - u =\ & \frac{du}{dx}\,r\ \cos\ \alpha + \frac{du}{dy}\,r\ \cos\ \beta + \frac{du}{dz}\,r\ \cos\ \gamma \\ +\ & \frac{1}{2}\frac{d^2u}{dx^2}\,r^2\ \cos^2\ \alpha + \frac{1}{2}\frac{d^2u}{dy^2}\,r^2\ \cos^2\ \beta + \frac{1}{2}\frac{d^2u}{dz^2}\,r^2\ \cos^2\ \gamma \\ +\ & \frac{d^2u}{dx\ dy}\,r^2\ \cos\ \alpha\ \cos\ \beta + \frac{d^2u}{dx\ dz}\,r^2\ \cos\ \alpha\ \cos\ \gamma + \frac{d^2u}{dy\ dz}\,r^2\ \cos\ \beta\ \cos\ \gamma \\ \cdots\ & . \end{aligned}$$

If we develop similarly R in accordance with the power and the products of $u'-u,\ x'-x,\ y'-y,\ z'-z$, we will have also

$$R = V + \left(\frac{dR}{du'}\right)(u'-u) + \left(\frac{dR}{dx'}\right)(x'-x) + \left(\frac{dR}{dy'}\right)(y'-y) + \left(\frac{dR}{dz'}\right)(z'-z) + \cdots\ ;$$

where, the parentheses indicating here that it needs to put $u' = u,\ x' = x,\ y' = y\ ,\ z' = z$ according to the differentiation which supposes r invariable, and V designating this which comes at the same time from the function Φ of the (no. 45), so that we have

$$V = \Phi\ (r,\ u,\ u,\ x,\ y,\ z,\ x,\ y,\ z). \tag{8}$$

(Below, original citation omitted.)

(§49.) (General equation of the motion of heat) [6]

In this hypothesis, we will stop the development of R at the terms dependent on the square of r exclusively. By reason of the system of R in respect to u and u', x and x', y and y', z and z', and of this one which V represents, we have evidently

$$\left(\frac{dR}{du'}\right) = \frac{1}{2}\frac{dV}{du}, \quad \left(\frac{dR}{dx'}\right) = \frac{1}{2}\frac{dV}{dx}, \quad \left(\frac{dR}{dy'}\right) = \frac{1}{2}\frac{dV}{dy}, \quad \left(\frac{dR}{dz'}\right) = \frac{1}{2}\frac{dV}{dz}\ ;$$

then, it will result hence

$$R = V + \frac{1}{2}\frac{dV}{du}\,(u'-u) + \frac{1}{2}\frac{dV}{dx}\,(x'-x) + \frac{1}{2}\frac{dV}{dy}\,(y'-y) + \frac{1}{2}\frac{dV}{dz}\,(z'-z)\ ;$$

and of this value jointed to that of $u'-u$, we will conclude

$$\begin{aligned} H_2 =\ & \frac{1}{2}\left[V\frac{d^2u}{dx^2} + \left(\frac{dV}{du}\frac{du}{dx} + \frac{dV}{dx}\right)\frac{du}{dx}\right]\int \cos^2\alpha\ ds + \frac{1}{2}\left[V\frac{d^2u}{dy^2} + \left(\frac{dV}{du}\frac{du}{dy} + \frac{dV}{dy}\right)\frac{du}{dy}\right]\int \cos^2\beta\ ds \\ +\ & \frac{1}{2}\left[V\frac{d^2u}{dz^2} + \left(\frac{dV}{du}\frac{du}{dz} + \frac{dV}{dz}\right)\frac{du}{dz}\right]\int \cos^2\gamma\ ds, \end{aligned}$$

[6]($\Downarrow$) This article is the most frequently referred from other article, such as 52, 58, 64, 68, 70, **76**, 85, 89, 117, 119, 120, 137, **162**. (These are the article numbers, referred to the no. 49, and in the bold numbers, the another equations are expressed.)

or more simply

$$
\begin{aligned}
H_2 &= \frac{1}{2}\Big[V\frac{d^2u}{dx^2}+\frac{dV}{dx}\frac{du}{dx}\Big]\int\cos^2\alpha\ ds+\frac{1}{2}\Big[V\frac{d^2u}{dy^2}+\frac{dV}{dy}\frac{du}{dy}\Big]\int\cos^2\beta\ ds\\
&+ \frac{1}{2}\Big[V\frac{d^2u}{dz^2}+\frac{dV}{dz}\frac{du}{dz}\Big]\int\cos^2\gamma\ ds\ ;
\end{aligned}
$$

the partial differences [7] of V with respect to $x,\ y,\ z$, being taken in considering u as a function of these three coordinates, and without varying r.

We have additionally

$$\int\cos^2\alpha\ ds=\int\cos^2\beta\ ds=\int\cos^2\gamma\ ds.$$

Moreover, if we call ψ the angle which makes the plane of the segment MM' and of a parallel to the axis of x traced through the point M, with a fixed plane traced through this parallel, we will have $ds=\sin\ \alpha\ d\alpha\ d\psi$; and the integral relative to ds will come to be extended to all the spherical surface, to which this element belongs, then it will result

$$\int\cos^2\alpha\ ds=\int_0^\pi\cos^2\alpha\ \sin\ \alpha\ d\alpha\int_0^{2\pi}d\psi=\frac{4\pi}{3}.$$

[8] Hence, in reducing the value of $\int R\ (u'-u)$ at the primary term $H_2\ r^2$ of the series (??), the equation (7) will come to be

$$
\begin{aligned}
c\frac{du}{dt} &= \frac{2\pi}{3}\Big(\frac{d^2u}{dx^2}\int_0^l V\ r^2\ dr+\frac{du}{dx}\int_0^l\frac{dV}{dx}\ r^2\ dr\Big)+\frac{2\pi}{3}\Big(\frac{d^2u}{dy^2}\int_0^l V\ r^2\ dr+\frac{du}{dy}\int_0^l\frac{dV}{dy}\ r^2\ dr\Big)\\
&+ \frac{2\pi}{3}\Big(\frac{d^2u}{dz^2}\int_0^l V\ r^2\ dr+\frac{du}{dz}\int_0^l\frac{dV}{dz}\ r^2\ dr\Big). \qquad (9)
\end{aligned}
$$

The function V being zero for all the value of r longer than l, we will be able to now extend the integral relative to r beyond this limit, and if we want to be until $r=\infty$. If we put also

$$\frac{2\pi}{3}\int_0^\infty Vr^2\ dr\equiv k, \qquad (10)$$

where, k will be a function of $u,\ x,\ y,\ z$, and we will have

$$\frac{2\pi}{3}\int_0^\infty\frac{dV}{dx}\ r^2\ dr=\frac{dk}{dx},\quad \frac{2\pi}{3}\int_0^\infty\frac{dV}{dy}\ r^2\ dr=\frac{dk}{dy},\quad \frac{2\pi}{3}\int_0^\infty\frac{dV}{dz}\ r^2\ dr=\frac{dk}{dz}\ ;$$

in consequence, the general equation of the motion of the heat will come to be finally [9]

$$(7)_{PS4}\qquad c\frac{du}{dt}=\frac{d.k\frac{du}{dx}}{dx}+\frac{d.k\frac{du}{dy}}{dy}+\frac{d.k\frac{du}{dz}}{dz}. \qquad (11)$$

When all the point of A gets to a stationary state, we will have $\frac{du}{dt}=0$, and then it will result

$$\frac{d.k\frac{du}{dx}}{dx}+\frac{d.k\frac{du}{dy}}{dy}+\frac{d.k\frac{du}{dz}}{dz}=0,$$

---

[7] (⇓) id.

[8] (⇓) According to [9, p.41, no.277],

$$\int\cos^m x\ \sin x\ dx=-\frac{\cos^{m+1}x}{m+1}.$$

[9] (⇓) The expression (9) is reduced into

$$c\frac{du}{dt}=\Big(\frac{d^2u}{dx^2}k+\frac{du}{dx}\frac{dk}{dx}\Big)+\Big(\frac{d^2u}{dy^2}k+\frac{du}{dy}\frac{dk}{dy}\Big)+\Big(\frac{d^2u}{dz^2}k+\frac{du}{dz}\frac{dk}{dz}\Big)\quad\Rightarrow\quad(11).$$

for the equation relative to this stationary state.
(§50.) The equation (11) coincides with that which I found in years ago for the case of a heterogeneous corps[10], however, in never supposing hence that the quantity k depended on the temperature u.

If A is a corps heterogeneous,

- k will depend only on u,
- and the equation (11) will be changed as follows : [11]

$$(8)_{PS4} \qquad c\frac{du}{dt} = k\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right) + \frac{dk}{du}\left(\frac{du^2}{dx^2} + \frac{du^2}{dy^2} + \frac{du^2}{dz^2}\right). \tag{12}$$

In supposing that this quantity k were independent of u, we could have the equation

$$(9)_{PS4} \qquad c\frac{du}{dt} = k\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right), \tag{13}$$

[12] which we give it ordinarily, and which is reduced, in the case of the stationary state, to an equation independent of two quantities c and k, viz.,

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} = 0. \tag{14}$$

[13]

Poisson puts also the another heat equations :

$$(1)_{PS11} \qquad \frac{du}{dt} = a^2\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right), \qquad \frac{k}{c} = a^2, \tag{15}$$

where, u is the heat, k and c are the conductibility and the specific heat of the material.

4. Conclusions

Fourier doesn't show the pricise deduction of the heat equation (1), while Poisson takes 9 pages to descrive it from §44 to §50. The diffefrence between Fourier and Poisson is the common kernel function of molecular distance, which Poisson considers and manipulates in both fluid motion and heat motion. Additionally, we concludes as follows :

- He presents the 'two constant theory', which we assert, (id. cf. [7]) as visible in the Navier-Stokes equations in 1831. After this, Stokes follows Poisson's equation in 1849, and Prandtl [18, 19] declears these equations as the 'Navier-Stokes equations' in the top of the twenty century.
- He presents the heat kernel function in [16], which is equivalent, however more complex than the fluid kernel function with the distance of fluid molecule : $f(r)$.
- He proposes the alternative method of the definite integral, [14] instead of making the universal method of it, since by Euler, Lagrange and Laplace.
- He shows the heat equation by deducing precisely, although Fourier's series is the first, however, its introduction isn't deducing such as Poisson's or without demonstration.
- Although his approach dues to the rivarly to the Fourier's theory, it brings up the derivative productions of the another solutions or thinking in making many breakthroughs to Fourier's method.

---

[10]sic. *Journal de l'École Polytechnique*, 19^e cahier, page 87. ($\Downarrow$) Poisson [11], [15, p. 677].

[11]($\Downarrow$) The second term of the right hand-side of the equation (12) is for $k = k(u)$, in sic.

[12]($\Downarrow$) The equation (13) means $c\frac{du}{dt} = k\Delta u$, where Δ meaning the Laplacian.

[13]($\Downarrow$) This function u satisfying the equation (14) is called harmonic function. Poisson doesn't mention the harmonic function, however, Poincaré [17, p.237] calls it so.

[14]($\Downarrow$) cf. [10], [15, pp.347-367].

5. Epilogue

Poisson [16, pp.411-415] expects the earth warming before the Industorial Revolution [15] up to 17 years after. According to his speculation, in using this average rate of the increment per a year is $0.22°C$, then we can estimate with this increment rate up to this year 2015, just at the COP21, the temperature rises between 198 years, $2.447°C$ as follows :

$$\frac{11.950 - 11.730}{17\frac{7}{12}} \times 198 = \frac{0.22}{17.58333} \times 198 = 2.477°C.$$

This is what is called the reason of the consensus about the increment of the earth warming in the world.

References

[1] J.M.C. Duhamel, (Book review) *Mémoire sur l'Équilibre et le Mouvement des Corps élastiques ; par M.Poisson*, Bulletin des sciences mathématiques, astromatiques, physiques et chimiques, **11**(1829), 98-111. (The title number : No.35.)

[2] J.-B.-J. Fourier, *Théorie analytique de la chaleur. Deuxième Édition*, Paris, 1822. (This is available by G.Darboux [?] [Tome Premier] with comments).

[3] J.-B.-J. Fourier, *Théorie analytique de la chaleur.*, Paris, 1822. → http://gallica.bnf.fr/ark:/12148/bpt6k1045508v (Remark. This is the original book version of BnF : *Bibliothèque Nationale France*, namely, it seems the original book of [2] edited by G.Darboux. After p.32, there are 4 pages of pp.23-26, however, p.33ff are followed after p.26. Total pages are 639 plus the 2 pages of figures, while the book [2] is consisted of 563 pages.)

[4] J.-B.-J. Fourier, *Mémoire d'Analyse sur le mouvement de la chaleur dans les Fluides*, pp.507-514, *Extrait des notes manuscrites conservées par l'auteur*, pp.515-530 Mémoires de l'Académie royale des Siences, **12**(1833), 507-530. (Lu : 4/sept/1820.) → http://gallica.bnf.fr/ark:/12148/bpt6k3227s/f620 (Darboux 2[?], pp.593-614)

[5] S. Masuda, *Historical development of classical fluid dynamics*, Scholars' Press, 2014.

[6] S. Masuda, *Historical development of classical physico-mathematics*, Scholars' Press, 2014.

[7] S. Masuda, *Confusion and Unity in Handling of Heat Motion and Fluid Motion in the 19th Century*, RIMS Kôkyûroku 1946, Kyoto univ. 2015/4.

[8] C.L.M.H. Navier, *Lettre de M.Navier à M.Arago*, Annales de chimie et de physique, **39**(1829), 99-107. (This is followed by *Note du Rédacteur*, 107-110.)

[9] B. O. Peirce, *A Short Table of Integrals, 4th ed.*, Brain Publishing, 1956. (Jap.)

[10] S.D. Poisson, *Mémoire sur les intégrales définies*, (1813), J. École Polytech., Cahier **16**, **9**(1813), 215-246. → http://gallica.bnf.fr/ark:/12148/bpt6k4336720/f220

[11] S.D. Poisson, *Mémoire sur la Distribution de la Chaleur dans les Corps solides*, J. École Royale Polytech., Cahier **19**, **12**(1823), 1-144. (Lu : 31/déc/1821.) → http://gallica.bnf.fr/ark:/12148/bpt6k433675h

[12] S.D. Poisson, *Mémoire sur les équations générales de l'équilibre et du mouvement des corps solides élastiques et des fluides*, (1829), J. École Royale Polytech., **13**(1831), 1-174. (Lu : 12/oct/1829.)

[13] S.D. Poisson, *Nouvelle théorie de l'action capillaire*, Bachelier Pére et Fils, Paris, 1831. → http://gallica.bnf.fr/ark:/12148/bpt6k1103201

[14] S.D. Poisson, *Traité de Méchanique*, second edition, (1), Bachelier, Imprimeur-Libraire, Paris, 1833. → http://gallica.bnf.fr/ark:/12148/bpt6k9605452x

[15] S.D. Poisson, *Traité de Méchanique*, second edition, (2), Bachelier, Imprimeur-Libraire, Paris, 1833. → http://gallica.bnf.fr/ark:/12148/bpt6k9605438n

[16] S.D. Poisson, *Théorie mathématique de la chaleur*, Bachelier Pére et Fils, Paris, 1835. → http://www.e-rara.ch/doi/10.3931/e-rara-16666

[17] E. Poincaré, *Théorie analytique de la propagation de la chaleur. Leçons professées pendant le premier semestre 1893-1894*, Georges Carré, Paris. → http://gallica.bnf.fr/ark:/12148/bpt6k5500702f

[18] L.Prandtl, *Über Flüssigkeitsbewegung bei sehr kleiner Reibung*, in III. Internationaler Mathematiker-Kongress in Heidelberg vom 8. bis 13. August 1904. *Verhundlungen*, A. Krazer (ed.) 184-91, Leibzig, 1905. Also : Ludwig Prandtl, *Gesammelte Abhundlungen zur Mecchanik, Hydro-und Aerodynamik*, vols **3**(1961), Göttingen. vol **2**, 575-584. (read 1904.)

[19] L. Prandtl, *Fundamentals of hydro-and aeromechanics*, McGrawhill, 1934. (Based on lectures of L.Prandtl (1929) by O.G.Tietjens, translated to English by L.Rosenhead. 1934.)

Remark. Lu : accepted date, (ex. Lu : 12/oct/1829, in the bibliographies of French Mémoire.)

---

[15](⇓) As we know, the Industorial Revolution was occurred at about in 1830 in England.